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ON LINEAR GROUPS OVER A FIELD OF FRACTIONS OF A POLYCYCLIC GROUP RING

BY

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ABSTRACT

Let G be a torsion free polycyclic-by-finite group and D be the field of fractions of the group algebra KG. Then any periodic subgroup of D_n is locally finite. This answers a question posed by D. Farkas.

Let G be a poly-⟨infinite cyclic⟩ group, C be a field and Δ be the (skew) field of fractions of the group ring CG.

D. Farkas posed in [3] a question whether periodic subgroups of the matrix group Δ_n are locally finite. His question was raised in connection with D. Segal's article [8], where the local finiteness of some periodic groups of automorphism is proven; in particular, these are subgroups of $(ZG)_n$, where G is polycyclic-by-finite.

The positive answer to Farkas' question follows from the results of this article.

Let D be a (skew) field. Consider a subgroup G of the multiplicative group D^* and a central subring C. We denote by $C(G)$ the subfield generated by C and G.

Our main result is (see corollary of Theorem 2 below):

Let $D = C(G)$ be a field generated by a polycyclic-by-finite group G. Then any periodic subgroup of D_n is locally finite.

In order to formulate Theorems 1 and 2 of the article we need some concepts from [6].

Let R be a ring, $S \subseteq R$ be a subring with the same unit. We remind one that a system of elements $e_1 = 1, e_2, \cdots, e_n$ is a normalizing basis of R over S if

(i) there holds for every $s \in S$

$$e_i s = \varphi_i(s) e_i, \tag{1.1}$$

where φ_i is an automorphism of S $(i = 1, 2, \cdots, n)$;

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(ii) the elements e_i $(i = 1, 2, \cdots, n)$ form a basis of R as a left S-module.

The group Φ of automorphisms of S, generated by all the automorphism φ_i in (1.1), will be called the automorphism group, generated by the basis e_i $(i = 1, 2, \cdots, n)$.

If now G is a subgroup of the group of units S^* of S then we say that Φ is almost inner on G if G is Φ invariant and all the elements of Φ, whose restrictions are inner on G, form a subgroup Φ_0 of finite index in Φ.

Our main result follows as a corollary of the following Theorem (see Theorem 2):

Let $D = K(G)$, where G is polycyclic-by-finite, $K = Z$ or Z_p, and R be a ring which has a basis e_i $(i = 1, 2, \cdots, n)$ which normalizes D and generates an almost inner group of automorphisms of G. Then any periodic subgroup of R^* is locally finite.

Indeed, if $R = D_n$, where D is generated by a polycyclic-by-finite group G, then the system of matrix units e_{ij} $(1 \leqq i \leqq n; 1 \leqq j = n)$ gives a normalizing basis of R over D which induces a unit group of automorphisms in G.

We consider also a case when the subgroup $F \subsetneq R^*$ consists of unipotent elements, i.e. $(f - 1)$ is nilpotent for every $f \in F$, and prove that under these conditions F must be nilpotent of class less than or equal to n; moreover, the subring T generated by all the elements $(f - 1)$, $f \in F$, is nilpotent of class less than or equal to n (see Theorem 1).

2. The Approximation Theorem and its corollaries

We denote in this section by D a field, generated by a polycyclic-by-finite group G over the subring K, generated by 1, and let R be a ring which has a basis which normalizes D and generates an almost inner group of automorphism in G. It is not difficult to show that G contains a characteristic poly-{infinite cyclic} subgroup H of finite index such that $H/\rho(H)$ is free abelian, where $\rho(H)$ is the nilpotent radical of H, and R has a basis $u_1 = 1, u_2, \cdots, u_n$ which normalizes H and a subfield $D_0 = K(H)$ and generates an almost inner group of automorphisms in H (see [6], lemma 3.1). As usual, we denote by $h(G)$ the Hirsch number of G.

We have under this assumption theorem 4.1 of [6].

Let nonzero elements

$$x_j \ (j = 1, 2, \cdots, m) \tag{2.1}$$

be given. Then a subring $Q \subsetneq R$ can be found such that

(i) $x_j \in Q$ $(j = 1, 2, \cdots, m)$ *and* $K[G] \subseteq Q$;

(ii) *there exists an epimorphism* θ *of* Q *on a ring* $\bar{Q}$ *of finite characteristic* p *with* $\ker\theta$ *a quasiregular ideal in* Q *such that* $\bar{Q}$ *contains a poly-{infinite cyclic} subgroup* $\bar{F}$, *where* $h(\bar{F}) \geqq 1$ *provided that* G *is not abelian-by-finite*; *the group* $\bar{F}$ *and* Z_p *generate in* $\bar{Q}$ *the group ring* $Z_p\bar{F}$;

(iii) $\bar{Q}$ *contains the field of fractions* Δ *of* $Z_p\bar{F}$ *and has a normalizing basis* $v_1 = 1, v_2, \cdots, v_l$ *over* Δ, *which generates an almost inner group of automorphism in* $\bar{F}$;

(iv) $\varphi(x_j) \neq \bar{0}$ $(j = 1, 2, \cdots, m)$;

(v) *an element* $q \in Q$ *is invertible in* Q *if and only if its image* $\bar{q} = \theta(q)$ *is invertible in* $\bar{Q}$;

(vi) *the Jacobson radical* $J(\bar{Q})$ *of* $\bar{Q}$ *satisfies the relation* $(J(\bar{Q}))^r = \bar{0}$.

We describe here the main steps in constructing the ring Q and the homomorphism θ; this is essential for the proof of Proposition 1 below.

Let Φ be the group of automorphisms, generated by the basis $u_1 = 1, u_2, \cdots, u_n$, and let Ψ be the group of automorphisms of H, generated by Φ together with the group $\operatorname{Inn} H$ of the inner automorphisms of H. If $g_1 = 1, g_2, \cdots, g_s$ be a transversal of H in G then one can adjoin, if necessary, these elements to the system of elements (2.1) and we assume therefore that the system (2.1) contains a transversal of H in G.

We have representations

$$x_j = \sum_{\alpha=1}^{n} x_{j\alpha} u_\alpha\,; \qquad x_{j\alpha} \in D_0 \quad (f = 1, 2, \cdots, m) \tag{2.2}$$

and

$$u_{\alpha_1} u_{\alpha_2} = \sum_{\alpha=1}^{n} y^{(\alpha)}_{\alpha_1,\alpha_2} u_\alpha, \qquad y^{(\alpha)}_{\alpha_1,\alpha_2} \in D_0 \quad (1 \leqq \alpha, \alpha_1, \alpha_2 \leqq n). \tag{2.3}$$

Consider now the subring $K[H]$ of D_0, generalized by H. Since H is polycyclic, $K[H]$ is a Noetherian domain and hence the field $D_0 = K(H)$ is a field of fractions of $K[H]$. We can therefore write out all the elements $x_{j\alpha}$, $y^{(\alpha)}_{\alpha_1,\alpha_2}$ in (2.2) and (2.3) in a form (c_β, d_β), $c_\beta \in K[H]$, $d_\beta \in K[H]$ $(\beta = 1, 2, \cdots, L)$.

Zalesskii's results on ideal correspondence in group rings of solvable groups ([7], 11.4) imply that $K[H]$ is isomorphic to the cross product

$$K[H] \simeq K[N] * H/N,$$

where N is the center of the Fitting subgroup $\rho(H)$ of H (see [6], corollary of proposition 2.3).

Since N is a free abelian group of finite rank and $K = Z$ or $K = Z_p$, any maximal ideal of $K[N]$ has a finite index; this implies that the Ψ-orbit of any maximal ideal is finite. It has been proven in [6] (see proposition 2.6 of [6]) that the results of Bergman and Roseblade related to Hall's problem on polycyclic groups allow one to find a maximal ideal $A \subseteq K[N]$ with a Ψ-orbit $A_1 = A, A_2, \cdots, A_k$, such that

$$c_\beta \notin A_i H, \quad d_\beta \notin A_i H \qquad (i = 1,2,\cdots,k;\ \beta = 1,2,\cdots,L).$$

It is not difficult to prove ([6], lemma 4.1) that H contains a characteristic subgroup U of finite index which stabilizes every ideal A_i $(i = 1,2,\cdots,k)$. Let

$$M = K[U] \setminus \bigcup_{i=1}^{k} A_i U = K[U] \setminus \bigcup_{i=1}^{k} A_i * U/N.$$

The set M is Ψ-invariant and it is proven in [6] (see the proof of theorem 4.1) that M is a right denominator set of regular elements in $K[H]$.

Consider the ring of fractions $K[H]_M$. Since D_0 is the ring of fractions of $K[H]$ we obtain that $K[H]_M \subseteq D_0$. This implies that the $K[H]_M$ module

$$Q = K[H]_M u_1 + K[H]_M u_2 + \cdots + K[H]_M u_n$$

is free on $u_1, u_2, \cdots, u_n$.

It is proven in [6] that Q is a subring of R containing all the elements x_α $(\alpha = 1,2,\cdots,m)$.

Finally, let

$$B = \bigcap_{i=1}^{k} A_i\,; \qquad (B) = BQ. \tag{2.4}$$

Then B is a Ψ-invariant ideal of $K[N]$, (B) is a quasiregular ideal of Q and $(B) = \ker \theta$. □

We need the following property of the ideal $\ker \theta = (B)$:

PROPOSITION 1. *The ideal $(B) = BQ$ is residually nilpotent*:

$$\bigcap_{s=1}^{\infty} (\mathrm{B})^s = 0. \tag{2.5}$$

PROOF. Since $K[N]$ is a Noetherian domain we obtain by the Krull theorem that

$$\bigcap_{s=1}^{\infty} B^s = 0.$$

Since U stabilizes every ideal A_i $(i = 1, 2, \cdots, k)$ and contains N we obtain that

$$A_iK[U] = A_iU = A_i * U/N \qquad (i = 1, 2, \cdots, k)$$

and

$$BK[U] = BU = B * U/N. \tag{2.6}$$

The relation (2.6) implies now that the ideal $BK[U]$ is residually nilpotent.

On the other hand we have the relation

$$B(K[U]) = (B * U/N)_M,$$

and it can be verified easily that the residual nilpotence of the ring $B * U/N$ implies that $(B * U/N)_M$ is residually nilpotent too.

If now $h_1 = 1, h_2, \cdots, h_r$ is a transversal of U in H then it gives a basis of $K[H]$ over $K[U]$, which normalizes $K[U]$ (see [6], lemma 4.1); it will be therefore a normalizing basis of $K[H]_M$ over $K[U]_M$ and it normalizes $(B * U/N)_M$ because B is H-invariant. Since the system of elements u_α $(\alpha = 1, 2, \cdots, n)$ is a normalizing basis of Q over $K[H]_M$ we obtain easily that the system

$$u_\alpha h_j \qquad (\alpha = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, r) \tag{2.7}$$

is a normalizing basis of Q over $K[U]_M$; it normalizes the ideal $(B * U/N)_M$ because B is Ψ-invariant.

This easily implies that $B(K[U]_M)Q = BQ$ is an ideal in Q and BQ has a normalizing basis (2.7) over $B(K[U]_M)$. Moreover, the residual nilpotence of $B(K[U]_M)$ implies that BQ is residually nilpotent. □

3. The proofs of the main results

Throughout this section D denotes a field, generated by a polycyclic-by-finite group G over Z or Z_p and R will be a ring, which has a basis $e_1 = 1, e_2, \cdots, e_n$ which normalizes D and generates an almost finite group ϕ of automorphisms in D.

We prove under these assumptions the following results.

PROPOSITION 2. *Let T be a finitely generated subring of R such that for any element $x \in T$ a representation*

$$x = \sum x_i \tag{3.1}$$

can be found such that all the elements in the right side of (3.1) *are nilpotent. Then* $T^n = 0$.

THEOREM 1. *Let F be an unipotent subgroup of R^* and T be the subring of R, generated by all the elements of the form $f-1$, $f \in F$. Then*

$$T^n = 0 \tag{3.2}$$

and, hence, $j_n(F) = 1$.

THEOREM 2. *Let F be a periodic subgroup of R^*. Then F is locally finite.*

Our proofs are based on the Approximation Theorem and use induction by the Hirsch number $h(G)$ of G. When $h(G) = 0$, G will be finite and R will be a finite dimensional algebra; Proposition 2 and Theorems 1 and 2 become in this case well-known classical theorems. This establishes the truth of the first step of the induction, when $h(G) = 0$, and we will deal therefore only with the second step.

PROOF OF PROPOSITION 2. First of all Theorem 30 of chapter 4 in [5] implies easily that any nilsubring in R is nilpotent of index less than or equal to n. It is enough therefore to prove that T is nil.

Take the element x in the left side of (3.1). Since $\dim_l(R:D) \leqq n$ the elements $1, x, \cdots, x^n$ are linearly dependent over D and we have therefore some relation

$$\lambda_1 x^{n_1} + \lambda_2 x^{n_2} + \cdots + \lambda_k x^{n_k} = 0, \tag{3.3}$$

where $0 \neq \lambda_j \in D$ $(j = 1, 2, \cdots, k)$ and $n \geqq n_1 > n_2 > \cdots > n_k \geqq 0$.

Prove that $x^{n_k} = 0$. Indeed, if $x^{n_k} \neq 0$ we can find by the Approximation Theorem a subring Q, which contains all the generators of T, the elements

$$\lambda_j, \lambda_j^{-1} \quad (j = 1, 2, \cdots, k); \quad x^{n_k}$$

and such that

$$\bar{x}^{n_k} \neq \bar{0} \tag{3.4}$$

in $\bar{Q}$. The relation (3.3) implies in $\bar{Q}$ a relation

$$\bar{\lambda}_1 \bar{x}^{n_1} + \bar{\lambda}_2 \bar{x}^{n_2} + \cdots + \bar{\lambda}_k \bar{x}^{n_k} = \bar{0}. \tag{3.3'}$$

Since Q contains all the generators of T, we have $T \subseteq Q$ and $\bar{T} \subseteq \bar{Q}$; hence $\bar{T}$ is nilpotent by the induction assumption and hence x is nilpotent.

Let s be the smallest natural number such that

$$\bar{x}^s = \bar{0}.$$

We will prove that $s = n_k$. This contradicts (3.4) and would therefore complete the proof.

Assume thus that $s > n_k$. Multiply (3.3′) by $\bar{x}^{s-n_k-1}$ and obtain

$$\bar{\lambda}_k \bar{x}^{s-1} = \bar{0},$$

which gives, via the fact that $\bar{\lambda}_k$ is invertible in $\bar{Q}$, that

$$\bar{x}^{s-1} = \bar{0},$$

which contradicts the definition of s. □

COROLLARY. *Let $X \subseteq R$ be a nilsemigroup. Then X is nilpotent of index less than or equal to n.*

PROOF. Consider any finitely generated subsemigroup $X_1 \subseteq X$. The subring, generated by X, satisfies the conditions of Proposition 1 and, hence, $X_1^n = 0$ and the assertion follows.

PROOF OF THEOREM 1. Let $F_1 = \mathrm{gp}(f_1, f_2, \cdots, f_k)$ be any finitely generated subgroup of F. The vector space (over Z_p or Z), generated by all the elements $f-1$, $f \in F_1$, is a subring $T_1 \subseteq T$, every element of which is a sum of nilpotent ones. The identity

$$(xy-1) = (x-1) + (y-1) + (x-1)(y-1)$$

shows that T_1 is generated by the elements $(f_i - 1)$, $i = 1, 2, \cdots, k$. Hence $T_1^n = 0$ by Proposition 1; since F_1 is an arbitrary subgroup of F, we obtain that $T^n = 0$. □

COROLLARY. *Assume that* $\operatorname{char} D = p$ *and let $P \subseteq R^*$ be a P-group. Then $\gamma_n(P) = 1$.*

PROOF. Consider the subring $T \subseteq R$, generated by all the elements of the form $(f-1)$, $f \in P$. Since the element $(f-1)$ is nilpotent when f is a p-element, the assertion follows from Theorem 1. □

The proof of Theorem 2 will make use of the following assertion.

PROPOSITION 3. *Let nonzero elements of R, x_j $(j = 1, 2, \cdots, m)$, be given and Q be a subring of R, which satisfies properties* (i)–(vi) *of the Approximation Theorem. Then any element of finite order in the group* $(1 + \ker \theta)$ *is a p-element.* (Here p is the characteristic of $\bar{Q}$.)

PROOF. Let

$$(1+x)^q = 1, \tag{3.5}$$

where $x \in (B)$ and q is a prime number. Apply Proposition 1 and find s such that $x \in (B)^s \setminus (B)^{s+1}$. The relation (3.5) now implies

$$qx \in (B)^{s+1}. \tag{3.6}$$

Assume that $q \neq p$. We will prove that this, together with (3.6), implies $x \in (B)^{s+1}$, which contradicts the choice of s.

Indeed, if $\operatorname{char} Q = p$ then (3.6) implies at once that $x \in (B)^{s+1}$. If now $\operatorname{char} Q = 0$ then the fact that $\operatorname{char} \bar{Q} = p$ implies that $p \in (B)$ and this together with the relation $x \in (B)^s$ implies

$$px \in (B)^{s+1},$$

which together with (3.6) gives once again $x \in (B)^{s+1}$. □

PROOF OF THEOREM 2. We can assume that F has a finite system of generators $f_1, f_2, \cdots, f_r$ and prove that F is finite.

Case 1. Assume that $\operatorname{char} D = p$.

Apply the Approximation Theorem and find the subring Q, containing $f_1, f_2, \cdots, f_r$, and hence the group F. The image $\bar{F}$ of F in the quotient ring $\bar{Q}$ is finite by the induction hypothesis and the kernel of the homomorphism $F \to \bar{F}$ induced by θ is a periodic subgroup P of $1 + \ker \theta$. By Proposition 3, P must be a p-group. Since $F/P \simeq \bar{F}$ is a finite group, P is finitely generated and it follows from Corollary of Theorem 1 that P is finite and hence F is finite. The assertion is thus proven in the case when $\operatorname{char} D = p$.

Case 2. $\operatorname{char} D = 0$.

We will find two different prime numbers p_i such that F is a finite extension of a p_i-group ($i = 1, 2$); this would imply, of course, that F is finite.

First apply the Approximation Theorem to the system of elements $f_1, f_2, \cdots, f_r$ and find a subring Q_1 with properties (i)–(vi). Since $\bar{Q}_1$ is a ring of finite characteristic, say p_1, Proposition 3 implies, via the truth of the assertion in case 1, that F is a finite extension of a p_1-group.

Consider now a new system of elements, $p_1, f_1, f_2, \cdots, f_r$ and find a subring $Q_2 \subseteq R$, which contains this system and satisfies conclusions (i)–(vi) of the Approximation Theorem. It is important that the property (iv) implies that $\bar{p}_1 \neq \bar{0}$ in the ring $\bar{Q}_2$ and hence $\operatorname{char} \bar{Q}_2$ is some prime number $p_2 \neq p_1$. Once again, as above, obtain that F is a finite extension of a p_2-group and the assertion follows. □

Theorems 1 and 2 and Proposition 2 were established under the assumption that $D = K(G)$, where $K = Z$ or Z_p. Consider now the case when $D = C(G)$ is generated by a polycyclic-by-finite group G over a central subring C and $R = D_n$. For any finite number of elements $x_1, x_2, \cdots, x_m \in R$ we can now find a finitely generated subring $C_0 \subseteq C$ such that $x_i \in \Delta_n$, where $\Delta = C_0(G)$. If C_0 is generated by elements z_j $(j = 1, 2, \cdots, n)$, then the subgroup $G_0 = \mathrm{gp}\{G, z_1, z_2, \cdots, z_n\}$ of Δ^* is polycyclic-by-finite and $\Delta = K(G_0)$, where $K = Z$ or Z_p. The following corollary of the results of this paragraph now follows easily.

Corollary. *Let $\Delta = C(G)$, where G is polycyclic-by-finite and C is a central subring of Δ. Then the conclusions of Proposition* 2, *Theorem* 1 *and Theorem* 2 *hold in the ring $R = \Delta_n$.*

In particular, the group ring KG of a polycyclic-by-finite torsion free group G is a domain (see [1], [2], [4]) and, hence, has a field of fractions Δ, and we see that the results of the article are applied to the ring Δ_n.

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